Partial Differential Equation

Separation of Variables Technique

Laplace's Equation in 2 dimensions (Cartesian co-ordinates) :

$$\nabla^2 V(x, y) = \partial^2V/\partial x^2 + \partial^2V/\partial y^2 = 0$$

Assume:V(x, y) is 'separable' in the product form : **V(x, y) = f(x) g(y).** Subst. in the diff. eqn. :

$$d^2f/dx^2\, g(y) + f(x)\, d^2g/dy^2 = 0.$$

Note that f(x) and g(y) are functions of single variables. So their derivatives are **not partial, but ordinary derivatives**. Divide both sides by V(x, y), i.e., f(x) g(y)..

$\Rightarrow\ f''(x)/f(x) + g''(y)/g(y) = 0$

$\Rightarrow\ f''(x)/f(x) = -g''(y)/g(y)$

Now, a function of 'x' cannot be equal to a function of 'y' for all values of x and y (they may, accidentally match at some particular pair of values of x and y), unless both are constant functions. [Note that **'$\phi(x)$ = constant' is a perfectly valid function.**] So, we conclude :

$f''(x)/f(x) = -g''(y)/g(y) = C$, where 'C' is called the **'separation constant'**.

If we chose the separation constant to be +ve, we shall have exponential solutions for f(x), but if we chose it to be –ve, we shall get sinusoidal (i.e., periodic) solutions. Suppose, we have the boundary conditions :

(i) V(x, y) = 0 at x = 0 for all values of y,

(ii) V(x, y) = 0 at x = a for all values of y.

This requires the solutions to repeat their values at x = 0 and x = a. So, we choose :

$\mathbf{C = -k^2}$ (i.e., -ve).

$\Rightarrow f''(x)/f(x) = -k^2,\ g''(y)/g(y) = +k^2$

$\Rightarrow$ $\mathbf{f(x) = A\cos kx + B\sin kx}$ and $\mathbf{g(y) = C\,e^{ky} + D\,e^{-ky}}$

$\Rightarrow$ **V(x, y) =** $[A\cos kx + B\sin kx]\,[C\,e^{ky} + D\,e^{-ky}]$

This is one solution for a particular value of 'k', but different values of 'k' will generate different solutions. The general solution is obtained by superposing them as :

$$\mathbf{V(x, y) = \Sigma_k\,[A_k\cos kx + B_k\sin kx]\,[C_k\,e^{ky} + D_k\,e^{-ky}]}$$

Note that the constants 'A_k', 'B_k', etc., may differ for different values of 'k'.

At x = 0, V = 0 for all values of y $\Rightarrow\ 0 = \Sigma_k A_k\,[C_k e^{ky} + D_k\,e^{-ky}]$

$\Rightarrow A_k = 0$

$\Rightarrow V(x, y) = \Sigma_k B_k \sin kx\,[C_k e^{ky} + D_k\,e^{-ky}]$

At x = a, V = 0 for all values of y $\Rightarrow$ either $B_k = 0$, or, sin ka = 0,

but both A_k and $B_k = 0$ will lead to the 'trivial solution' V(x, y) = 0 for all x and y.

So, we turn towards the other choice : sin ka = 0 $\Rightarrow$ ka = nπ, or, $\mathbf{k = n\pi/a}$**.**

We see, how the boundary condition can restrict the possible choices for 'k'.

Now, $V(x, y) = \Sigma_n B_n \sin(n\pi x/a)\,[C_n\,e^{(n\pi y/a)} + D_n\,e^{-(n\pi y/a)}]$.

We have replaced 'k' by(nπ/a) and re-parametrized the constants 'A_k', 'B_k', etc., as 'A_n', 'B_n', etc.

We may absorb the const. B_n in $C_n$and D_n , calling : $B_n C_n = C_n'$ and $B_nD_n = D_n'$, so that :

$$\mathbf{V(x, y) = \Sigma_n \sin(n\pi x/a)\,[C_n'\,e^{(n\pi y/a)} + D_n'\,e^{-(n\pi y/a)}].}$$

Suppose, we have another boundary condition : (iii) V(x, y) = 0 at y = 0 for all values of x.

This will imply : $0 = \Sigma_n \sin(n\pi x/a)\,[C_n' + D_n'] \Rightarrow D_n' = -\,C_n'$

$\Rightarrow$ **$V(x, y) = \Sigma_n \sin(n\pi x/a) \times C_n'[e^{(n\pi y/a)} - e^{-(n\pi y/a)}]$**

$= \Sigma_n 2C_n' \sin(n\pi x/a) \sinh(n\pi y/a)$,

where $\sinh(\theta)$ is dedfined as : $[e^{\theta} - e^{-\theta}]/2$ and $\cosh(\theta)$ as : $[e^{\theta} + e^{-\theta}]/2$.

We shall require another (fourth) boundary cond. to determine the const. C_n'.