Planck' Law

(Derivation without assuming Bose-Einstein Statistics)

Consider a closed chamber with reflecting walls within which some Electro-magnetic radiation is confined. The electric and the magnetic field will satisfy the wave equation :

$$\nabla^2\mathbf{E} = (1/c^2)\,\partial^2\mathbf{E}/\partial t^2,\quad \nabla^2\mathbf{H} = (1/c^2)\,\partial^2\mathbf{H}/\partial t^2$$

The equations can be solved by the 'separation of variable' technique. Any component of '**E**', or '**H**', say E_x, assumes the form :

$$E_x\,(x, y, z, t) = X(x)\,Y(y)\,Z(z)\,T(t),\quad \text{where}\quad X''/X + Y''/Y + Z''/Z = (1/c^2)\,T''/T$$

After separation : $X''/X = -k_1^2,\ \ Y''/Y = -k_2^2,\ \ Z''/Z - k_3^2$

$$\text{and}\quad (1/c^2)\,T''/T = -(k_1^2 + k_2^2 + k_3^2)$$

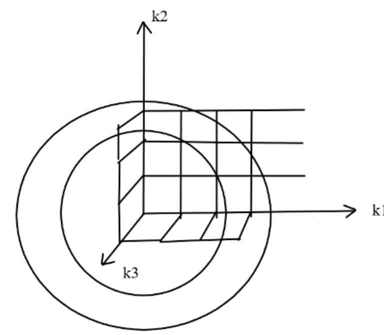
$$\Rightarrow\ T''/T = -c^2\,(k_1^2 + k_2^2 + k_3^2) = -\,c^2\,\mathbf{k}^2 = -\,\omega^2$$

With boundary conditions of the form : $E_x = 0$ for $x = 0$ and ℓ_1, $y = 0$ and ℓ_2, $z = 0$ and ℓ_3, where ℓ_1, ℓ_2, ℓ_3 are the dimensions of the chamber, the solutions for E_x will be of the form :

$$E_x = A_{mnp}\sin(m\pi x/\ell_1)\,\sin(n\pi y/\ell_2)\,\sin(p\pi z/\ell_3),$$

$$\text{where}\quad k_1^2 + k_2^2 + k_3^2 = m^2\pi^2/\ell_1^2 + n^2\pi^2/\ell_2^2 + p^2\pi^2/\ell_3^2 = \omega^2/c^2$$

Each set of values (k_1, k_2, k_3) represent a '**mode**' of vibration. Plotted in a 3-dimensional graph, they form a lattice, while the eqn. : ω = const., describes a sphere of radius $\boldsymbol{\omega/c_{\ell}}$. Number of modes within the frequency range $\omega \rightarrow \omega + d\omega$ equals the no. of lattice points within two concentric spheres.



Each cube in the lattice has **8** lattice points at its corners, but each corner point, in turn, belongs to **8** neighbouring cubes.

$\Rightarrow$ No. of lattice points /cube = **1**.

$\Rightarrow$ No. of lattice points within the frequency range $\omega \rightarrow \omega + d\omega$ = vol. of the spherical shell / vol. of a cube.

$\Delta m = \Delta n = \Delta p = 1 \ \Rightarrow\ \Delta k_1 = \pi/\ell_1,\ \Delta k_2 = \pi/\ell_2,\ \Delta k_3 = \pi/\ell_3$

$$\Rightarrow\ \textbf{vol. of a cube} = \boldsymbol{\pi^3/\ell_1\ell_2\ell_3}$$

and the **vol. of the spherical shell** $= 4\pi r^2\,dr = 4\pi\,(\omega^2/c^2)\,d\omega/c = \boldsymbol{4\pi\omega^2\,d\omega/c^3}$

$$= 4\pi(2\pi\nu)^2\,d(2\pi\nu)/c^3\ \ [\because\ \omega = 2\pi\nu] = 32\,\pi^4\nu^2\,d\nu/c^3$$

So, the no. of modes within $\nu \rightarrow \nu + d\nu = (32\,\pi^4\nu^2\,d\nu/c^3) \div (\pi^3/\ell_1\ell_2\ell_3) = 32\,\pi\nu^2 d\nu/c^3 \times \ell_1\ell_2\ell_3$

$\Rightarrow$ the no. of such modes **per unit vol. of the cavity** $= \boldsymbol{32\,\pi\nu^2 d\nu/c^3}$.

One must note that +ve and –ve values of (m, n, p) do ***not*** represent different modes, since the solution : $E_x = A_{mnp}\sin(m\pi x/\ell_1)\sin(n\pi y/\ell_2)\sin(p\pi z/\ell_3)$, only differs in sign, which can be absorbed in the const. A_{mnp}. Therefore, to obtain the no. of physically distinct modes, we must restrict ourselves to only **one octant** of the spherical shell

$$\Rightarrow\ \text{the no. of modes} = \boldsymbol{4\,\pi\nu^2 d\nu/c^3}.$$

Similar solutions can be obtained for the other components of the electric field. However, for a wave travelling, say along z-direction, we shall have only the E_x and the E_y components, since

Electromagnetic wave is transverse. The two components basically take care of the two independent polarizations (**E** along x, consequently, **H** along y and **E** along y, consequently, **H** along x). Taking these two polarization states into account :

the total no. of modes / unit vol. of the cavity = $\mathbf{8\pi\nu^2 d\nu / c^3}$.

Now each vibrational mode may be considered as an independent harmonic oscillator.

In the earlier note [Einstein Sp. Heat Problem], we calculated the average energy $\langle E_\nu \rangle$ for each oscillator with freq. 'ν' :

[The single-particle Partition Function :

$$z = \Sigma e^{-\beta E_n} = e^{-\beta h\nu/2} + e^{-3\beta h\nu/2} + e^{-5\beta h\nu/2} + \ldots$$

$$= e^{-\beta h\nu/2}(1 + e^{-\beta h\nu} + e^{-2\beta h\nu} + \ldots)'$$

This is an infinite GP series, with the first term = $e^{-\beta h\nu/2}$ and the common ratio = $e^{-\beta h\nu}$.

The sum of the series : $a + ar + ar^2 + \ldots = a/(1 - r)$

$$\Rightarrow \mathbf{z = e^{-\beta h\nu/2} / (1 - e^{-\beta h\nu})} \text{ ----- (1)}$$

$$\Rightarrow \ln z = -\beta h\nu/2 - \ln(1 - e^{-\beta h\nu})$$

$\Rightarrow$ Avg. energy per oscillator : $\langle E_\nu \rangle = -\partial \ln z/\partial\beta = h\nu/2 + 1/(1 - e^{-\beta h\nu}) \times e^{-\beta h\nu} \times h\nu$

$$\Rightarrow \langle E_\nu \rangle = -\partial \ln z/\partial\beta = h\nu/2 + h\nu\, e^{-\beta h\nu} / (1 - e^{-\beta h\nu})$$

The first term is clearly the **zero point energy**.

Multiplying the numerator and the denominator of the second term by $e^{+\beta h\nu}$:

$$\mathbf{\langle E_\nu \rangle = h\nu/2 + h\nu / (e^{\beta h\nu} - 1)} \text{]}$$

Dropping the zero point energy term, the avg. energy/unit vol. of the cavity, within the freq. range $\nu \to \nu + d\nu$:

$$\mathbf{u_\nu\, d\nu = 8\pi\nu^2 d\nu/c^3 \times h\nu / (e^{\beta h\nu} - 1),}$$

which is nothing but **Planck's law.**

Derivation of Planck's Law from Bose-Einstein Statistics

Instead of considering Electro-magnetic waves, travelling back and forth between the walls of a black-body chamber, we may consider the system to be a collection of photons, moving randomly, like the molecules of an ideal gas. The state of a photon may be specified by its position, momentum and state of polarization.

Within a small region $dx\, dy\, dz\, dp_x\, dp_y\, dp_z$ in the 6-dimensional phase space, the number of states is given by :

$$2 \times dx\, dy\, dz\, dp_x\, dp_y\, dp_z / h^3.$$

The factor '2' is due to the two states of polarization.

(As such, every point in the phase space **classically** represents a state of a particle. However, according to Uncertainty Principle, no two states within the hyper-cube (6-dimensional cube) of volume h^3 can be distinguished.)

Integrating over all possible values of x, y, z :

the number of states in the ranges : p_x to $p_x + dp_x$, p_y to $p_y + dp_y$, dp_z to $p_z + dp_z$ =

$$\int_x \int_y \int_z dx\, dy\, dz\, dp_x\, dp_y\, dp_z / h^3 = 2V\, dp_x\, dp_y\, dp_z / h^3,$$

where 'V' is the volume of the box within which the radiation is confined.
Switching over to the polar co-ordinates **in the momentum space** :

$$dp_x\, dp_y\, dp_z = p^2 \sin\theta\, dp\, d\theta\, d\phi,$$

where θ, ϕ specifies the direction of the momentum vector.
Integrating over all possible directions of the momentum vector (because the energy depends only on the magnitude of the **p**-vector) :
the number of states in the momentum (magnitude) range p to p + dp =

$$g(p)\, dp = 2V/h^3 \int_\theta \int_\phi p^2 \sin\theta\, dp\, d\theta\, d\phi = 8\pi p^2\, dp \times V/h^3.$$

<u>In terms of energy :</u>

The energy of a relativistic particle in general, is given by the expression :

$$E = \sqrt{(c^2p^2 + m_0{}^2c^4)},$$ where 'm_0' is the rest mass of the particle.

For a photon, $\mathbf{m_0 = 0 \Rightarrow E = cp} \Rightarrow dE = c\, dp$
So, the number of states in the energy range E to E + dE =

$$g(E)\, dE = 8\pi V/c^3h^3 \times E^2\, dE \times \sqrt{(2mE)}$$

<u>In terms of frequency :</u>

In accordance with Quantum Postulate, the energy of a photon is given by :
$\mathbf{E = h\nu}$. Hence, the number of states in the range of frequency ν to $\nu + d\nu$ =

$$g(\nu)\, d\nu = 8\pi V/c^3 \times \nu^2\, d\nu$$

Note that the expression is independent of 'h', which hints that it might have a **classical derivation**, which was indeed provided by **Raleigh and Jeans**.
According to **Bose Einstein Statistics**, the number of particles occupying a state with energy 'E', is given by : $\mathbf{1/e^{E/KT} - 1}$.
(Since the total number of photons need not be conserved, the chemical potential $\mu = 0$.)
So, the number of photons occupying the states within the frequency range ν to $\nu + d\nu$ =

$$n(\nu)\, d\nu = 8\pi V/c^3 \times \nu^2\, d\nu/\, (e^{h\nu/KT} - 1)$$

and the energy /unit volume within this range =

$$E(\nu)\, d\nu = 8\pi/c^3 \times \nu^2\, d\nu/\, (e^{h\nu/KT} - 1) \times h\nu = 8\pi h\nu^2\, d\nu/c^3(e^{h\nu/KT} - 1),$$

which is nothing but Planck's law.

<u>Total Energy Density :</u>

Considering all possible frequencies, the total Energy Density

$$u = \int 8\, \pi\nu^2 d\nu/c^3 \times h\nu\, /\, (e^{\beta h\nu} - 1).$$

Subst. : $\beta h\nu = h\nu\, /KT = x \Rightarrow (h/KT)\, d\nu = dx$

At $\nu = 0$, $x = 0$ and as $\nu \to \infty$, $x \to \infty$.

$$\Rightarrow \text{Now, } u = \int 8\, \pi h\nu^3 d\nu/c^3/\, (e^{\beta h\nu} - 1) \quad [\text{for } \nu = 0 \text{ to } \infty].$$

$$= 8\, \pi h\, /c^3\, (KT/h)^4 \int x^3 dx\, /\, (e^x - 1) \quad [\text{for } x = 0 \text{ to } \infty].$$

The x-integral produces a pure number, hence, $\mathbf{u \propto T^4}$. This is basically the **Stefan's Law**.